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## 1. The First Boundary Value Problem of Potential Theory.

Given an open continuum, $T$, in the plane of $x$ and $y$, and values, $F$, assigned to each of the boundary points of $T$, the first boundary value problem of potential theory consists in finding a function which shall have continuous second derivatives which satisfy LaPlace's equation at the points of $T$, and which shall approach the given values, $F$, on the boundary. For continuous boundary values, and for a broad class of regions $T$, the problem has been proven possible of solution.[1] We shall denote by $[T_1]$ the class of regions for which the problem is possible in this sense.

## 2. The Example.

The purpose of the present note is to give an example of a region $T_1$ to which previous proofs do not apply; and incidentally, to throw some light on the direction which further investigation may take.

We first recall a familiar point set obtained by the "removal of middle thirds." The set $S_0$ is the closed interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ of the $x$-axis. The set $S_1$ is obtained from $S_0$ by the removal of its open middle third, and consists, therefore, in the two closed intervals $\left(-\frac{1}{2}, -\frac{1}{6}\right)$ and

[1] The most general results of this class appear to be due to Lebesgue, Sur le problème de Dirichlet, Rendiconti di Palermo, v. 24 (1907), pp. 371–402. His restriction on $T$ is, that any point belongs to $T$ if it is possible to surround the point by a curve, lying in an arbitrarily small neighborhood of the point, and consisting only of points of $T$. Results more general, at least in some directions, have been obtained by H. B. Phillips and Norbert Wiener, and by G. E. Raynor. Their papers have appeared, and are expected to appear, in the *Journal of Mathematics and Physics of the Massachusetts Institute of Technology*, vol. 2 (1923), pp. 105–124, and in the *Annals of Mathematics*, respectively. For simply connected regions in particular, G. C. Evans has shown that the problem is solvable when the values F are merely bounded summable functions of the values on the boundary of the conjugate to a Green function, provided a suitable understanding be agreed on as to the manner of approach to the boundary values. See Problems of Potential Theory, *Proceedings of the National Academy of Sciences*, vol. 7 (1921), pp. 89–98.

$\left(\frac{1}{6}, \frac{1}{2}\right)$. $S_n$ is obtained from $S_{n-1}$ by the removal of the open middle thirds of the $2^{n-1}$ equal intervals of which $S_{n-1}$ consists. The set $S$ is the points common to all the sets $S_n$ as $n$ runs through all positive integers.

Let $K$ be a circle containing $S$ in its interior, say the circle $x^2 + y^2 = 4$. The region, $T^*$, in question, consists of the points interior to $K$, excluding the points of $S$. The boundary points of $T^*$ within $K$ then form a perfect set of Borel measure 0. But in spite of the relative sparseness of the points of $S$ and the infinite connectivity of $T^*$, this region belongs to the class $[T_1]$.

## 3. The Lebesgue Barrier Sets.

This assertion may be proved by means of the notion of barrier functions, introduced by Lebesgue.[2] Suppose that there exists, for each boundary point, $B$, of a given region, $T$, and for every positive number, $\epsilon$, a function, $U_{B,\epsilon}(P)$, with the following properties: it is harmonic in $T$, and assumes at every point, $P$, of $T$, a value not less than the distance $\overline{BP}$. As $P$ approaches $B$, the function approaches a number less than $\epsilon$, or more generally, there exists a neighborhood, $N$, of $B$, such that when $P$ is a point of $T$ in $N$, $0 \leqq U_{B,\epsilon}(P) \leqq \epsilon$. A set of such functions, we call, after Lebesgue, a barrier set for the region $T$. *Its existence is a necessary and sufficient condition that* T *belongs to* $[T_1]$. While this theorem, as well as its demonstration, is implied in the report of Lebesgue, the reader may welcome the following more explicit proof.

First, suppose that the barrier set exists for $T$. We approach the region $T$ by a set of regions, $t_1, t_2, t_3, \ldots$, each containing the preceding and bounded by a finite number of regular curves, and such that each point of $T$ is interior to one of them, and hence to all later ones. Let us suppose first that the boundary values $F$ are those of a polynomial, $F(x, y)$. Then harmonic functions exist for the regions $t_1, t_2, t_3, \ldots$, which approach the same values as $F$ on their boundaries. These functions will at least contain a sequence, $[U_n]$, which converges uniformly in any closed subregion of $T$ to a harmonic function $U$. That $U$ will approach the required boundary value at a boundary point, $B$, may be seen as follows. Let $L$ be an upper bound for the difference

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[2] Sur le problème de Dirichlet, C. R. vol. 154 (1912), p. 335.

quotients of $F$ in $T + T'$, $T'$, being the boundary set of $T$. Then, for any $n$, $U_n$ lies between $F(B) - LU_{B,\epsilon}(P)$ and $F(B) + LU_{B,\epsilon}(P)$, since its values on the boundary of $t_n$ do, and these functions are harmonic. Hence $U$ lies between these functions, for every positive $\epsilon$, and hence approaches $F(B)$.

If $F$ is not given as a polynomial, we first extend its definition so that it shall be continuous in the whole plane.[3] We may then approximate to $F$ by a sequence of polynomials over a closed region containing $T'$ in its interior, the approach being uniform. The corresponding harmonic functions of $T$, $U_1$, $U_2$, $U_3$, ... will then approach uniformly in $T$ a harmonic function with the required boundary values. If $T'$ extends to infinity, we have merely to replace, in the above reasoning, the polynomials by rational functions with a single pole not in $T'$.

Conversely, a barrier set exists for each region $T_1$, one, indeed, independent of $\epsilon$. For, to each boundary point, $B$, corresponds a harmonic function of $T_1$, $U_B(P)$, determined by the boundary values $F(P) = \overline{BP}$. To see that $U_B(P) - F(P)$ is never negative, we have only to recall Gauss' mean value theorem for harmonic functions, and to notice that the above difference, at any point, $P$, of $T$, exceeds its arithmetic mean on any circle in $T_1$ with center at $P$. Such a function can have no minimum in $T_1$, and as the present one vanishes on $T_1'$ and is continuous in $T_1 + T_1'$, it can, accordingly, never be negative in this region.

## 4. A Logarithmic Spread on $S$.

It remains to establish the existence of a barrier set for $T^*$. There is no difficulty connected with the barrier functions for the points $B$ of $K$, since a barrier for the open circle evidently is a barrier for $T^*$. As a preliminary step in the establishment of barriers for the points of $S$, we consider the logarithmic potential of a certain spread of attracting matter on $S$. It will simplify notation to shift the origin of coördinates for the present to the point $\left(-\frac{1}{2}, 0\right)$. The potential in question is then the Stieltjes integral $L(P) = \int_0^1 \log r \, d\mu(x)$, where

[3] For the possibility of this, see Lebesgue, l. c. Rendiconti di Palermo, or, Carathéorody, Vorlesungen über reele Funktionen, Leipzig, 1918, pp. 617–620.

$\mu(x)$ is the total mass on the segment $(0, x)$, and is defined as follows: $\mu(0) = 0$, $\mu(1) = 1$, on $\frac{1}{3} \leqq x \leqq \frac{2}{3}$, $\mu(x) = \frac{1}{2}$; in general, $\mu(x)$ being defined on the gaps in $S_n$ plus their end points (i.e. on $S_o - S_n$ plus its derivative), it is defined on the new gaps of $S_n$ and their end points (i.e. on $S_{n-1} - S_n$ plus its derivative) by assigning to it the arithmetic mean of its values at the two nearest points to right and left at which it has been defined. In this way it is determined at points everywhere dense, and the definition is completed by demanding that it be continuous, a thing evidently possible.

The function $L(P)$ is seen to be harmonic in the entire plane, apart from the points of $S$. Moreover, *it is continuous at the points of* S. To show this, we first determine a simple function dominating $\Delta\mu = \mu(x_2) - \mu(x_1)$, $(x_2 > x_1)$.

If $x_1 = 0$, then $\Delta\mu = \mu(x_2) = \frac{1}{2^n}$ when $x_2 = \frac{1}{3^n}$. If we eliminate $n$ between the equations $x = \frac{1}{3^n}$ and $y = \frac{1}{2^n}$, we obtain the function $y = x^p$ $(p = \log 2/\log 3)$. Geometric intuition suggests that $x^p$ dominates $\mu(x)$, and further, that $(\Delta\mu) \leqq (\Delta x)^p$, since $y = \mu(x)$ never increases more rapidly than at the origin. These inferences are correct. We first note that $y = \mu(x)$ never rises above the polygonal line through the points $\left(\frac{1}{3^n}, \frac{1}{2^n}\right)$ of $y = x^p$. For as $x$ increases from $\frac{1}{3^n}$ to $\frac{2}{3^n}$, $\mu(x)$ remains constant at $\frac{1}{2^n}$, and $\mu(x)$ does not increase beyond half way to $\frac{1}{2^{n-1}}$ until $x$ has progressed $\frac{2}{3}$ of the remaining distance from $\frac{2}{3^n}$ to $\frac{1}{3^{n-1}}$. Hence for $\frac{1}{3^n} \leqq x \leqq \frac{2}{3^n} + \frac{2}{3^{n+1}}$, $y = \mu(x)$ lies under the chord joining $\left(\frac{1}{3^n}, \frac{1}{2^n}\right)$ and $\left(\frac{1}{3^{n-1}}, \frac{1}{2^{n-1}}\right)$. But from $\frac{2}{3^n} + \frac{1}{3^{n+1}}$ to $\frac{2}{3^n} + \frac{2}{3^{n+1}}, + \frac{2}{3^{n+2}}$, by the same reasoning, $\mu(x)$ lies under its steeper chord joining $\left(\frac{2}{3^n} + \frac{1}{3^{n+1}}, \frac{1}{2^n} + \frac{1}{2^{n+1}}\right)$ and $\left(\frac{1}{3^{n-1}}, \frac{1}{2^{n-1}}\right)$. A continuation of the argument, and the use of the continuity of $\mu(x)$ completes the proof that $\mu(x)$ lies under the polygonal line in question, and hence under the concave downward curve $y = x^p$.

Next, suppose that $x_1$ is the abscissa of the left end of one of the

intervals of $S_n$. We are evidently at liberty to suppose that $n$ is the smallest integer for which $x_1$ has this property. Note that all the parts of $\mu(x)$ above the intervals of $S_n$ are congruent. Hence for $\Delta x \leqq \frac{1}{3^n}$, $\Delta\mu \leqq (\Delta x)^p$. That the same inequality holds for all $\Delta x$ with the above value of $x_1$ follows from the congruence of the pieces of $\mu(x)$, the constancy of $\mu(x)$ between the intervals of $S_n$, and from the fact that if we call $S_n'$ the set obtained by shifting $S_n$ to the right a distance $x_1$, *the total length of the intervals of $S_n$ between $x_1$ and $x_2$ is never greater than the total length of the intervals of $S_n'$ between $x_1$ and $x_2$.* To see this, we notice first that $S_n$ and $S_n'$ begin at $x_1$ with a common interval of length $\frac{1}{3^n}$. There follows a gap of length $\frac{1}{3^n}$ between the intervals of $S_n'$, whereas a greater gap, of length $g$, say, follows for $S_n$, since $n$ was the smallest integer for which $x_1$ began an interval of $S_n$. The corresponding gap $g$ in $S_n'$ must begin at a distance $g$ from $x_1$, i. e. a distance $\frac{1}{3^n}$ before the gap $g$ in $S_n$ terminates. Because of the symmetry of $S_n$ to either side of a gap at least for a distance $g$, it follows that after the gap $g$ in $S_n'$, the two sets have again a common interval, and that before this point the accumulated length of intervals of $S_n'$ has exceeded, or at least equaled that of $S_n$, because of the earlier occurrence of the gap $g$ in $S_n$. The reasoning may now be repeated. The next gap in $S_n'$ will be of length $\frac{1}{3^n}$, while the next gap in $S_n$ will be longer, for otherwise we should have intervals of $S_{n-1}$ and $S'_{n-1}$ coinciding, which is contrary to the hypothesis that $n$ is the smallest integer for which $x_1$ begins an interval of $S_n$.

Finally, if, instead of beginning an interval of $S_n$, $x_1$ lies in a gap between two of them, the inequality $\Delta\mu \leqq (\Delta x)^p$ holds *a fortiori*, since the diminution of $x_1$ will have increased $\Delta x$ without changing $\Delta\mu$. Because of the continuity of $\mu(x)$, the inequality holds generally.

The continuity of $L(P)$ now follows without trouble if we separate out from the integral a part corresponding to a small interval containing the boundary point $B$, in whose neighborhood the continuity of $L(P)$ is to be investigated. The above inequality shows that this interval can be made so small that the integral over it may be made uniformly less than a preassigned positive $\epsilon$ in a given neighborhood of $B$. The rest of the integral is continuous at $B$, and hence so is $L(P)$.

We now know that $L(P)$ attains its negative minimum, $-m$, on $S$.

It is also negative at all points within a circle $C$ with $S_0$ as diameter, for the integrand is never positive in this circle. On and outside any larger concentric circle, of radius $R$, $L(P) > \log\left(R - \frac{1}{2}\right)$.

## 5. The Barriers for $T^*$.

We now shift our origin of coördinates back to its position of symmetry in the set $S$, and denote by $U_1(x, y)$ the function which $L(P) + m$ thus becomes. We then consider the subset, $s$, of $S$, namely the points common to $S$ and one of the intervals, $i$, of $S_n$; let $(a, 0)$ be the center of this interval. Then $U_2(x, y) = U_1(a + 3^n x, 3^n y)$ is harmonic in the whole plane except at the points of $s$, it is never negative, and within the circle $c$, which has $i$ as diameter, $U_2(x, y) \leqq m$. On and without any circle with $(a, 0)$ as center, and radius $R/3^n$, $U_2(x, y) > \log\left(R - \frac{1}{2}\right)$. Or, with $3^n R$ in place of $R$, we may state that on and outside a circle of radius $R$ about $(a, 0)$, $U_2(x, y) > n\log 3 + \log\left(R - \frac{1}{2\cdot 3^n}\right)$. The least value possible of $R$ on $K$ is $\frac{3}{2} + \frac{1}{2\cdot 3^n}$. Hence, on $K$, $U_2(x, y) > n\log 3$.

Now let $B$ be any point of $S$. We identify the above subset $s$ with one containing $B$. The maximum of $\overline{BP}$ on $K$ is $5/2$. Hence $U_3(x, y) = \frac{5U_2(x, y)}{n 2\log 3}$ exceeds $\overline{BP}$ on $K$, is harmonic except on $s$, is never negative, but within $c$ is less than $5m/n2\log 3$. Finally, $U_{B,n}(x, y) = U_3(x, y) + \frac{1}{3^n}$ exceeds $\overline{BP}$ on $s$ and $K$, and hence in $T^*$. As $P$ approaches $B$, this function approaches a value which vanishes with $1/n$, and thus has all the properties of the barrier functions, so that the existence of the barrier set, and with it the fact that $T^*$ belongs to $[T_1]$ have been established.

## 6. Remarks.

The preceding considerations give an example of the possible usefulness of the notion of barrier. As the nature of the region $T$ which makes it a $T_1$ depends on the boundary "im Kleinen," a generalization of the idea of barrier may be indicated which may be useful, namely a

barrier for the region bounded by a large circle and an arbitrarily small piece of the boundary set.

As to the degree of generality of the regions $T_1$, the following may be said. An isolated point cannot form part of the boundary (see, for instance, Osgood, Funktionentheorie, p. 565). Hence the boundary must be dense on itself, and as it is closed, it must be perfect. The above developments suggest that a criterion as to which perfect sets may enter the boundary, may be found in the possibility of logarithmic spreads on the boundary set, of only positive masses, which have potentials that are continuous on the boundary.